

AUTOMORPHISMS OF THE GAUSSIAN UNIMODULAR GROUP

BY

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1. **Introduction.** Let G be the ring of Gaussian integers, and G_n the group of $n \times n$ unimodular matrices over G . Define $G_n^+ = \{X \in G_n: \det X = +1\}$, and likewise define G_n^-, G_n^i, G_n^{-i} . Let $X' =$ transpose of X , $\bar{X} =$ conjugate of X , $I^{(n)} =$ identity matrix in G_n , $0 =$ null matrix of appropriate size, and $A \dot{+} B =$ direct sum of A and B . For $X, Y \in G_n$, write $X \sim Y$ if X and Y are conjugate in G_n . We assume throughout that $n \geq 2$. For $a =$ unit in G_n and $A \in G_{n-1}$ define $(a) + {}^r A$ to be the matrix B for which $b_{rr} = a$, $b_{rj} = b_{jr} = 0$ for $j \neq r$, and such that the submatrix obtained by deleting the r th row and r th column from B coincides with A . Thus $(a) + {}^1 A$ coincides with the ordinary direct sum $(a) \dot{+} A$. We use $[a_1, \dots, a_n]$ to denote the diagonal matrix with diagonal elements $a_1, \dots, a_n$.

The generators of G_n are [1, p. 425]

$$(1.1) \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \dot{+} I^{(n-2)}, \quad S = \begin{pmatrix} 0 \dots 0 & (-1)^{n-1} \\ 1 \dots 0 & 0 \\ \dots & \dots \\ 0 \dots 1 & 0 \end{pmatrix}, \quad P = (i) \dot{+} I^{(n-1)}.$$

For the case $n = 2$ we shall use T_0, S_0, P_0 as symbols for the generators, where S_0 now denotes the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this paper we prove the following

MAIN THEOREM. *Let $\mathfrak{A}_n$ be the automorphism group of G_n . Then $\mathfrak{A}_n$ is generated by*

1. $X \rightarrow AXA^{-1}, A \in G_n;$
2. $X \rightarrow X'^{-1}$ (may be omitted when $n = 2$);
3. $X \rightarrow \bar{X};$
4. $X \rightarrow (\det X)^k X$, where $k = 1$ if n is even, and $k = 2$ if n is odd;
5. For $n = 2$ only, $(P_0, S_0, T_0) \rightarrow (P_0, -S_0, -T_0).$

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We may remark that 1., 2. and 3. are obviously automorphisms. We shall prove later (Lemmas 3.2 and 3.3) that 4. and 5. are also automorphisms. For $n=2$, it is easily verified that 2. is expressible in terms of the other automorphisms in the list, hence may be omitted.

2. Involutions in G_n . We begin by giving a canonical form for involutions in G_n under conjugacy. Throughout this paper let ξ denote $1+i$, and set

$$(2.1) \quad J_\alpha = \begin{pmatrix} -1 & 0 \\ \alpha & 1 \end{pmatrix},$$

$$(2.2) \quad W(a, b, c, d) = (J_1 \dot{+} \cdots \dot{+} J_1) \dot{+} (J_\xi \dot{+} \cdots \dot{+} J_\xi) + (-I)^{(c)} + I^{(d)},$$

where a J_1 's and b J_ξ 's occur in (2.2).

THEOREM 2.1. *As (a, b, c, d) range over all non-negative integers for which $2a+2b+c+d=n$, the matrices $W(a, b, c, d)$ give a full set of non-conjugate involutions in G_n .*

Proof. The proof given in [2, pp. 336–337] can be used with a few modifications, due to the fact that G is a principal ideal ring. From the reasoning there, it is easily established that for any involution $X \in G_n$, we have

$$(2.3) \quad X \sim \begin{pmatrix} -I^{(q)} & 0 \\ T & I^{(p)} \end{pmatrix}$$

where T is a diagonal matrix with entries 0, 1 or ξ . The right-hand side of (2.3) is conjugate to some $W(a, b, c, d)$, and it is not hard to verify that two distinct $W(a_j, b_j, c_j, d_j)$ ($j=1, 2$) cannot be conjugate in G_n .

We may remark that p, q in (2.3) are the dimensions of the plus-space X^+ , and the minus-space X^- , respectively, of the involution X . Call X a (p, q) involution in such case. We find at once that $W(a, b, c, d)$ is an $(a+b+d, a+b+c)$ involution.

Our next step will be to characterize the $\pm(1, n-1)$ involutions in G_n . Let $\mathfrak{C}(S)$ denote the centralizer in G_n of a set S of elements in G_n .

LEMMA 2.1. *Let $X \in G_n$ be an involution and let*

$$\mathfrak{M} = \{ M \in \mathfrak{C}(X) : M \equiv X \pmod{2} \}.$$

Then the only involutions in $\mathfrak{C}(\mathfrak{M})$ are $\pm I^{(n)}$, $\pm X$.

Proof. For fixed $B \in G_n$ we note that $M \in \mathfrak{M}$ implies $BMB^{-1} \in \mathfrak{C}(BXB^{-1})$ and $BMB^{-1} \equiv BXB^{-1} \pmod{2}$, and conversely. Without loss of generality, we may therefore take X in the form of the right-hand side of (2.3). In that case, $\mathfrak{C}(X)$ consists of all elements $K \in G_n$ given by

$$(2.4) \quad K = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad A \in G_q, \quad D \in G_p, \quad C = (DT - TA)/2.$$

Since $A \equiv -I \pmod{4}$ and $D \equiv I \pmod{4}$ imply $C \equiv T \pmod{2}$ we see that $\mathfrak{M}$ contains all elements K satisfying

$$(2.5) \quad A \equiv -I \pmod{4}, \quad D \equiv I \pmod{4}.$$

Now if $Y \in \mathfrak{C}(\mathfrak{M})$, then Y commutes with all K satisfying (2.5). Set

$$Y = \begin{pmatrix} Y_1 & 0 \\ Y_2 & Y_3 \end{pmatrix}.$$

In that case, Y_1 commutes with all $A \in G_q$ for which $A \equiv -I \pmod{4}$, and Y_3 commutes with all $D \in G_p$ for which $D \equiv I \pmod{4}$. This shows at once that $Y_1 = u \cdot I$, $Y_3 = v \cdot I$, u, v units. If, further, Y is to be an involution, it follows that $Y_1 = \pm I$, $Y_3 = \pm I$. Since $Y \in \mathfrak{C}(\mathfrak{M})$ implies $Y \in \mathfrak{C}(X)$, therefore $Y_2 = (Y_3 T - T Y_1)/2$, and Y_2 is uniquely determined by Y_1 and Y_3 . Hence $\mathfrak{C}(\mathfrak{M})$ contains at most four involutions.

THEOREM 2.2. *The image of any $(1, n-1)$ involution in G_n under any $\tau \in \mathfrak{A}_n$ must be either a $(1, n-1)$ or an $(n-1, 1)$ involution.*

Proof. The result is trivial for $n=2$ and $n=3$. Assume hereafter that $n>3$. We shall characterize the $\pm(1, n-1)$ involutions in G_n by intrinsic properties using a method due to Mackey [5]; see also Rickart [7]. Letting $\mathfrak{C}^2(\cdots)$ denote $\mathfrak{C}(\mathfrak{C}(\cdots))$, define for an involution $X \in G_n$

$$(2.6) \quad \nu(X) = \text{Max (number of involutions in } \mathfrak{C}^2(X, X_1)),$$

where X_1 ranges over all involutions in $\mathfrak{C}(X)$. Taking X to be a (p, q) involution we shall show that $\nu(X) \geq 16$ if $\text{Min}(p, q) > 1$, while $\nu(X) = 8$ if $\text{Min}(p, q) = 1$; this will imply the theorem.

To begin with, note that $\nu(X)$ depends only upon the conjugate class of the involution X . We may therefore take X as the right-hand side of (2.3), and then $\mathfrak{C}(X)$ is given by all K satisfying (2.4). For $\text{Min}(p, q) > 1$ define

$$(2.7) \quad X_1 = \begin{bmatrix} -1 & & & \\ & I & & \\ & & -1 & \\ & & & I \end{bmatrix}, \quad W = \begin{bmatrix} 1 & & & \\ & I & & \\ -t & & -1 & \\ & & & I \end{bmatrix}$$

where we have set T (occurring in (2.3)) $= (t) + T_1$. Then both X_1 and W are involutions in $\mathfrak{C}(X)$. Since every $K \in \mathfrak{C}(X_1)$ is of the form

$$K = \begin{bmatrix} a & b & & \\ & A_1 & B_1 & \\ c & & d & \\ & C_1 & D_1 & \end{bmatrix},$$

we see that the general element of $\mathfrak{G}(X, X_1)$ is given by

$$(2.8) \quad K = \begin{pmatrix} a & & 0 & \\ & A_1 & & \\ c & & d & \\ & C_1 & & D_1 \end{pmatrix},$$

where

$$(2.9) \quad c = (d - a)t/2, \quad C_1 = (D_1T_1 - T_1A_1)/2.$$

But then the involution W defined above commutes with all such K , so that $\mathfrak{G}^2(X, X_1)$ contains the 16 distinct involutions $\pm X^a X_1^b W^c$, ($a, b, c = 0, 1$), and $\nu(X) \geq 16$ for $\text{Min } (p, q) > 1$. (Indeed, $\nu(X) = 16$ for this case although we do not need the stronger result here.)

We now show that $\nu(X) = 8$ for $p = 1$. We may choose

$$(2.10) \quad X = \begin{pmatrix} -1 & 0 \\ t & I^{(n-1)} \end{pmatrix},$$

$t = (t, 0, \dots, 0)'$. Then $\mathfrak{G}(X)$ consists of all elements

$$(2.11) \quad K = \begin{pmatrix} a & 0 \\ c & D \end{pmatrix}, \quad a \in G_1, \quad D \in G_{n-1}, \quad c = (D - aI)t/2.$$

To compute $\nu(X)$ we may assume that $X_1 \in \mathfrak{G}(X)$ is given by

$$(2.12) \quad X_1 = \begin{pmatrix} 1 & 0 \\ u & U \end{pmatrix}, \quad u = (U - I)t/2,$$

where $U \in G_{n-1}$ is an involution. Then $\mathfrak{G}(X, X_1)$ consists of all K given by (2.11) for which $D \in \mathfrak{G}(U)$. In particular, whenever $D \in \mathfrak{G}(U)$ and $D \equiv U \pmod{2}$ then (2.11), with $a = 1$, defines an element of $\mathfrak{G}(X, X_1)$. If now $L \in \mathfrak{G}^2(X, X_1)$ is an involution, then L has the form

$$L = \begin{pmatrix} a^* & 0 \\ c^* & D^* \end{pmatrix}, \quad c^* = (D^* - a^*I)t^*/2$$

and L commutes with every K for which $a = 1$, $D \in \mathfrak{G}(U)$ and $D \equiv U \pmod{2}$. Then D^* commutes with all such D , whence by Lemma 2.1, $D^* = \pm I$ or $\pm U$. Certainly $a^* = \pm 1$. Since a^* and D^* uniquely determine c^* it follows that $\mathfrak{G}^2(X, X_1)$ contains at most 8 involutions. Since $\pm I, \pm X, \pm X_1, \pm XX_1$ are all in $\mathfrak{G}^2(X, X_1)$ we have established that $\nu(X) = 8$ if $\text{Min } (p, q) = 1$.

Let us now set

$$L_\alpha = J_\alpha \dot{+} I^{(n-2)}, \quad \alpha = 0, 1 \text{ or } \xi.$$

Then every $(1, n-1)$ involution in G_n is conjugate to one of L_0, L_1, L_ξ , and

hence any $\tau \in \mathfrak{A}_n$ maps L_0 onto $\pm AL_\alpha A^{-1}$ for some $A \in G_n$, where $\alpha = 0, 1$ or ξ .

THEOREM 2.3. *For $\tau \in \mathfrak{A}_n$ there exists $A \in G_n$ such that $L_0^\tau = \pm AL_0 A^{-1}$.*

Proof. For $n \geq 3$ we shall use the method of maximal sets of involutions (see [6]). By a *maximal set* in G_n we mean an abelian group of 2^n involutions in G_n . As in [6] we may at once establish the following results:

(i) The number of elements in any abelian group of involutions is $\leq 2^n$.
(ii) A maximal set contains precisely $C_{n,p}$ involutions of type (p, q) .
(iii) Any maximal set may be obtained from a *generating matrix* M^n (whose columns are primitive vectors with components in G) by choosing any p columns of M as basis for the plus-space W^+ of an involution, and the remaining q columns as basis for W^- . Each such choice defines a unique involution W , and this process gives rise to $C_{n,p}$ involutions of type (p, q) . If this process is carried out for $p = 0, 1, \dots, n$, an abelian group of 2^n involutions is obtained. Furthermore, if each of the invariant factors of M is either 1, ξ or 2, then each of the 2^n involutions will lie in G_n . In this case we call M a *permissible generating matrix*.

(iv) Two permissible generators M_1, M_2 give rise to conjugate maximal sets if and only if there exist $A, B \in G_n$, where B is obtained from I by permuting columns and multiplying them by units, such that $M_2 = A M_1 B$. In such case call M_1, M_2 *equivalent*.

(v) Every permissible generating matrix is equivalent to one of the form

$$(2.13) \quad M^{(n)} = \begin{pmatrix} I^{(r)} & A & B \\ 0 & \xi I^{(s)} & C \\ 0 & 0 & 2I^{(t)} \end{pmatrix},$$

where the columns of M are primitive, the elements of A are 0's and 1's, those of B are 0, 1 or ξ , and those of C are 0 or ξ .

Now define M_1 by: $s = 1, t = 0$, all entries in A are 1's; M_2 by: $r = 1, t = 0$, all entries in A are 1's; M_3 by: $s = 0, t = 1$, all entries in B are 1's; M_4 by: $r = 1, s = 0$, all entries in B are 1's. The maximal sets generated by M_1 and M_2 are nonconjugate (since $n \geq 3$), and each contains n involutions which are conjugate to L_ξ . The maximal sets generated by M_3 and M_4 are nonconjugate, and each contains n involutions conjugate to L_1 .

On the other hand, it is easy to show that any two maximal sets, each of which contains n involutions conjugate to L_0 , must be conjugate. Hence for $n \geq 3$ the class of L_0 is characterized by intrinsic properties, and the theorem holds. We postpone until later the proof for $n = 2$.

3. General remarks. Before we turn to the question of determining all automorphisms of G_n , it is desirable to state several lemmas.

LEMMA 3.1. *For any automorphism τ of G_n , either $\det X^\tau = \det X$ for all $X \in G_n$ or $\det X^\tau = \text{conjugate of } \det X$ for all $X \in G_n$.*

Proof. Let $S^{(k)} = P^{-k}SP^k$, $T^{(k)} = P^{-k}TP^k$. Since every $X \in G_n$ is expressible as a power product of P , S and T we find that every X can be written as

$$X = P^m \Pi(S, T, S^{(k)}, T^{(k)}),$$

and then $\det X = i^m$. Exactly as in [2, Corollary 1 of Theorem 1], we deduce that $\det S^r = \det T^r = 1$, whence also $\det (S^{(k)})^r = \det (T^{(k)})^r = 1$. Hence we have

$$\det X^r = (\det P^r)^m.$$

But $\det P^r = \pm i$, since if $\det P^r = \pm 1$, then $G_n^r \subset (G_n^+ \cup G_n^-)$, which is impossible. Hence $\det X^r = (\pm i)^m$, where $\det X = i^m$, whence the result follows.

LEMMA 3.2. *For n even the mapping $X \rightarrow (\det X)X$ is an automorphism of G_n . For n odd $X \rightarrow (\det X)^2 X$ is an automorphism of G_n .*

Proof. Consider first the case where n is even. The mapping is clearly an endomorphism of G_n . If $X^r = I$, then $(\det X)X = I$ whence $X = u \cdot I$, $u = (\det X)^{-1}$. But then $\det X = u^n$, so $u^n = u^{-1}$, whence $u = 1$ (because n is even). Therefore τ is one-to-one.

To show that τ is onto, we observe that $S^r = S$, $T^r = T$; set $Q = -iP$ for $n \equiv 0 \pmod{4}$, and $Q = iP$ for $n \equiv 2 \pmod{4}$. In either case $Q^r = P$, whence τ is onto.

A similar proof is valid for odd n .

LEMMA 3.3. *For $n = 2$, the mapping τ defined by $P_0^r = P_0$, $S_0^r = -S_0$, $T_0^r = -T_0$ is an automorphism of G_2 .*

Proof. To begin with, we must show that τ induces a well-defined mapping of G_2 into itself. This will be so if we can show that if a power product $\prod(P_0, S_0, T_0) = I$ in G_2 then the total number of factors of S_0 and T_0 is even.

Letting $\xi = 1 + i$ as usual, we remark that since

$$P_0 \equiv I \pmod{\xi}, \quad \prod(P_0, S_0, T_0) = I$$

implies

$$(3.1) \quad \prod(S_0, T_0) \equiv I \pmod{\xi}.$$

However, there are only 6 elements in $G_2 \pmod{\xi}$, represented by I , S_0 , T_0 , S_0T_0 , T_0S_0 , $S_0T_0S_0$, since $S_0^2 \equiv T_0^2 \equiv I \pmod{\xi}$. Any power product $\prod(S_0, T_0)$ can be brought into one of these 6 forms by repeated use of $S_0^2 \equiv T_0^2 \equiv (S_0T_0)^3 \equiv I \pmod{\xi}$. Hence in the left-hand side of (3.1), the total number of S_0 's and T_0 's must be even.

Now that τ has been shown to be well-defined we see at once that τ is onto. Further $\tau^2 = 1$ implies τ is one-to-one, whence τ is indeed an automorphism of G_2 .

4. **Generators of $\mathfrak{A}_2$.** We shall obtain here the generators of $\mathfrak{A}_2$, the automorphism group of G_2 . As before, define

$$J_\alpha = \begin{pmatrix} -1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

Our previous discussion shows that $\pm I$ and J_α , $\alpha=0, 1, \xi$ constitute a full set of nonconjugate involutions in G_2 .

LEMMA 4.1. *For any $\tau \in \mathfrak{A}_2$, there exists an $A \in G_2$ such that $J'_1 = AJ_1A^{-1}$.*

Proof. By Theorem 2.2, to within an inner automorphism we have $J'_1 = \pm J_\alpha$, $\alpha=0, 1$ or ξ . However, the centralizer $\mathfrak{C}(J_1)$ contains 8 elements, whereas $\mathfrak{C}(J_\alpha)$ contains 16 elements for $\alpha=0$ or ξ . This completes the proof since $-J_1$ is conjugate to J_1 .

THEOREM 4.1. $\mathfrak{A}_2$ is generated by the automorphisms

1. $X \rightarrow AXA^{-1}$,
2. $X \rightarrow X'^{-1}$,
3. $X \rightarrow \bar{X}$,
4. $X \rightarrow (\det X)X$,
5. $(P_0, S_0, T_0) \rightarrow (P_0, -S_0, -T_0)$.

Proof. Let $\tau \in \mathfrak{A}_2$; changing τ by an inner automorphism if necessary, we may assume hereafter that $J'_1 = J_1$. Let $K = S_0J_0$, then

$$(4.1) \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad KJ_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (KJ_1)^3 = -I, \quad K^2 = I.$$

Let us put

$$K^\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using the fact $J'_1 = J_1$, (4.1) implies that

$$-a + b + d = 1, \quad b(a + d) = c(a + d) = 0, \quad a^2 + bc = d^2 + bc = 1.$$

These imply that $d = -a$, $b = 2a + 1$, and

$$a^2 + (2a + 1)c = 1,$$

that is

$$4(a + c)^2 - (2c - 1)^2 = 3.$$

There are only 4 solutions in Gaussian integers of this equation, and **therefore** K^τ has only 4 possible expressions given by K, K_1, K_2, K_3 where

$$K_1 = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad K_3 = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}.$$

A further inner automorphism by a factor of J_1 leaves J'_1 unaltered, but **takes**

K_2 into K , and K_3 into K_1 . We may therefore assume from now on that $J_1^\tau = J_1$ and either $K^\tau = K$ or $K^\tau = K_1$. In the latter case, replace τ by the automorphism

$$X \rightarrow (V^{-1}X^\tau V)^{-1}, \quad \text{where } V = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

This new automorphism leaves J_1 and K invariant. Hence in all cases, after changing τ by automorphisms chosen from the list in Theorem 4.1, we may assume that $J_1^\tau = J_1$ and $K^\tau = K$.

LEMMA 4.2. *If $\tau \in \mathfrak{A}_2$ is such that $J_1^\tau = J_1$ and $K^\tau = K$, then $J_0^\tau = \pm J_0$.*

Proof. We have $J_0K = -KJ_0$, $J_0^2 = I$. Setting

$$J_0^\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have $d = -a$, $c = -b$, $a^2 - b^2 = 1$. Solving this last equation in Gaussian integers, we find that there are only 4 possibilities for J_0^τ , namely $\pm J_0$ or $\pm iS_0$.

Suppose now that $J_0 = \pm iS_0$. Since $J_0 = P_0^2$, setting

$$P_0^\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we obtain

$$\begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \pm \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Hence $a+d$ is a unit. On the other hand,

$$a^2 + d^2 = -2bc, \quad (a+d)^2 = 2(ad - bc),$$

so $(a+d)^2$ is a nonunit. This is a contradiction, whence J_0^τ must be $\pm J_0$.

We have now shown that by changing the given τ by automorphisms in the list, we can assume that

$$J_0^\tau = \pm J_0, \quad J_1^\tau = J_1, \quad K^\tau = K.$$

CASE I. $J_0^\tau = J_0$. Then $S_0 = KJ_0$ implies $S_0^\tau = S_0$, and $T_0 = KJ_0J_1K$ implies $T_0^\tau = T_0$. From $P_0^2 = J_0$, setting

$$P_0^\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$a^2 + bc = -1, \quad d^2 + bc = 1, \quad b(a + d) = c(a + d) = 0.$$

These imply that $b=c=0$, and $a=\pm i$, $d=\pm 1$. Hence $P_0^\tau = \pm P_0$ or $\pm \bar{P}_0$. In the latter case, changing τ by $X \rightarrow \bar{X}$, we may assume $P_0^\tau = \pm P_0$. This new τ is the automorphism $X \rightarrow (\det X)^m X$ where $m=4$ or 2 , and hence is a product of automorphisms on the list.

CASE II. $J_0^\tau = -J_0$. As above, we find that $S_0^\tau = -S_0$, $T_0^\tau = -T_0$, $P_0^\tau = \pm iP_0$ or $\pm i\bar{P}_0$. In the latter case, change τ by $X \rightarrow \bar{X}$ to obtain

$$S_0^\tau = -S_0, \quad T_0^\tau = -T_0, \quad P_0^\tau = \pm iP_0.$$

This automorphism is an obvious product of automorphisms on the list.

This completes the proof of Theorem 4.1. We may remark that $X \rightarrow X'^{-1}$ can be omitted from the list, since it can be expressed as a product of the other automorphisms on the list. Further, Theorem 4.1 implies Theorem 2.3 for the case $n=2$.

5. **Generators of $\mathfrak{A}_3$.** In this section we prove the main theorem for the case $n=3$.

STEP 1. Let D_j be obtained from $I^{(3)}$ by changing the j th diagonal element to -1 . Given any automorphism $\tau \in \mathfrak{A}_3$, we may assume by Theorem 2.3 (after changing τ by an inner automorphism) that $D_1^\tau = \pm D_1$. In that case τ maps $\mathfrak{C}(D_1)$ onto itself, that is,

$$(5.1) \quad \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}^\tau = \begin{pmatrix} b & 0 \\ 0 & B \end{pmatrix}$$

where a is a unit in G , $A \in G_2$. By Lemma 3.1 $\tau: G_3^+ \rightarrow G_3^+$ so that $\tau: \mathfrak{C}(D_1) \cap G_3^+ \rightarrow \mathfrak{C}(D_1) \cap G_3^+$. For each $A \in G_2$ choose a to be a unit for which $a \nmid A \in G_3^+$. Then b and B in (5.1) are uniquely determined by A . Set

$$(5.2) \quad \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}^\tau = \begin{pmatrix} \lambda(A) & 0 \\ 0 & A^\sigma \end{pmatrix}, \quad \text{where } a \cdot \det A = 1, A \in G_2.$$

Then $\lambda: G_2 \rightarrow G_1$ is a homomorphism, as is $\sigma: G_2 \rightarrow G_2$. Since $\lambda(A) \cdot \det A^\sigma = 1$ we see that if $A^\sigma = I$ then $\lambda(A) = 1$, and so $A = I$. Hence σ is one-to-one, and from this we see that σ is an automorphism of G_2 . Consequently $\det A^\sigma = \det A$ always or conjugate of $\det A$ always, whence $\lambda(A) = a$ always or $\bar{a}$ always. Therefore $\lambda(A) = 1$ for $A \in G_2^+$ and $\lambda(A) = -1$ for $A \in G_2^-$.

Using the results of the preceding section, we deduce that there exists a $Y \in G_2$ such that

$$A^\sigma = (\det A)^m Y A^* Y^{-1}$$

for all $A \in G_2$ where A^* is obtained from A by applying neither, or one, or both of automorphisms 3., 5. (§4). If we change τ by an inner automorphism with a factor of $1 \nmid Y^{-1}$, we may then assume that $A^\sigma = (\det A)^m A^*$, and that $D_1^\tau = \pm D_1$ is still valid.

Let us apply the above results to evaluate D_j^τ , $j=2, 3$. We have

$$(D_1 D_2)^\tau = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^\tau = \begin{pmatrix} \lambda(A) & 0 \\ 0 & A^\sigma \end{pmatrix},$$

where $A = (-1) \dot{+} (1)$. Then $\lambda(A) = -1$ by the above remarks since $A \in G_2^-$. Further, $A = P_0^2$, so $A^* = A$ for any choice of $*$. Therefore $A^\sigma = \pm A$, whence $(D_1 D_2)^\tau = D_1 D_2$ or $D_1 D_3$. The latter case is reduced to the former by changing τ by an inner automorphism with factor $(1) \dot{+} K$ where K is given in (4.1). Therefore we obtain $D_1^\tau = \pm D_1$, $D_2^\tau = \pm D_2$. Since $(-I)^\tau = -I$ we also have $D_3^\tau = \pm D_3$. Thus, starting with any $\tau \in \mathfrak{A}_3$ and changing τ by inner automorphisms, we arrive at a new τ for which $D_j^\tau = \pm D_j$, $j=1, 2, 3$.

STEP 2. Now let $\tau \in \mathfrak{A}_3$ satisfy $D_r^\tau = \pm D_r$, where $r=1, 2, 3$. By the preceding discussion we may set

$$(5.3) \quad ((a) +^\tau A)^\tau = (\lambda_r(A) +^\tau A^\sigma)^\tau,$$

where $A \in G_2$ is arbitrary, a is a unit such that $a \cdot \det A = 1$, $\lambda_r: G_2 \rightarrow G_1$ is a homomorphism such that either $\lambda_r(A) = a$ for all $A \in G_2$ or $\lambda_r(A) = \bar{a}$ for all $A \in G_2$, and $\sigma_r \in \mathfrak{A}_2$ is expressible as

$$(5.4) \quad A^{\sigma_r} = (\det A)^{m_r} Y_r A^{\omega_r} Y_r^{-1}, \quad \text{for all } A \in G_2,$$

where $Y_r \in G_2$, and A^{ω_r} is obtained by applying to A neither, or one, or both of automorphisms 3. and 5. (§4).

Now we evaluate $((-1) \dot{+} A)^\tau$ where $A = (-1) \dot{+} (1)$. By the above this yields

$$Y_1 A Y_1^{-1} = \pm A$$

whence Y_1 is either diagonal or anti-diagonal, that is

$$Y_1 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \text{or} \quad Y_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

u and v units. A similar argument shows that each Y_r is either diagonal or anti-diagonal.

CASE I. Suppose to begin with that at least one Y_r is diagonal; without loss of generality we may assume that Y_1 is diagonal. After an inner automorphism with factor $(1) \dot{+} Y_1^{-1}$ we may assume that $Y_1 = I$ in (5.4); Y_2 and Y_3 will now be different, but $D_r^\tau = \pm D_r$ is still valid. We may again deduce that Y_3 is either diagonal or anti-diagonal.

CASE I(a). Suppose Y_3 is diagonal, say $Y_3 = [u, v]$. Then changing τ by an inner automorphism with factor $[u^{-1}, v^{-1}, v^{-1}]$ we still have $Y_1 = I$, $D_r^\tau = \pm D_r$, and now also $Y_3 = I$. Therefore

$$T^\tau = (T_0 + (1))^\tau = T_0^{\omega_3} \dot{+} (1),$$

where now $T_0^{\omega_3} = \pm T_0$; the minus sign occurs if and only if automorphism 5. (§4) is one of the factors of ω_3 . We show next that $T_0^{\omega_3} = -T_0$ is impossible.

For, if $T^\tau = -T_0 \dot{+} (1)$, then $(S_0 \dot{+} (1))^\tau = -S_0 \dot{+} (1)$. Set

$$(5.5) \quad U = ((1) \dot{+} S_0) \cdot (S_0 \dot{+} (1)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $U^\tau = ((1) \dot{+} (\pm S_0))(-S_0 \dot{+} (1)) = U_1$ or U_2 , according to the sign, where

$$U_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Set $Z = TU^2$; then

$$(5.6) \quad T' = T'_0 \dot{+} (1) = (UZ^{-1})^2 UZ^2$$

and $T'_0 = S_0^{-1}T_0S_0$. Therefore $(T'_0)^{\omega_3} = -T'_0$. Applying τ to both sides of (5.6), and using $U^\tau = U_1$ or U_2 we obtain a contradiction. Hence $T_0^{\omega_3} = -T_0$ cannot occur.

We may now assume that both T and $S_0 \dot{+} (1)$ are invariant under τ . In that case, again defining U by (5.5), U^τ has the two possible values U , U_3 , where $U_3 = D_3UD_3$. But $S = U^2$ so that either $S^\tau = S$ or $S^\tau = D_3SD_3$.

In order to find P^τ , we observe that

$$V = D_1D_2 \cdot iP = [1, -i, i] = (1) \dot{+} V_1$$

where $V_1 = [-i, i]$; hence $V = (1) \dot{+} V_1^{\sigma_1}$. But $V_1 = P_0^3S_0^{-1}P_0S_0$, whence $V_1^{\sigma_1} = V_1$ or $\bar{V}_1$. Using the fact that $(iI)^\tau = \pm iI$ we obtain $P^\tau = \pm P$ or $\pm \bar{P}$. In the latter case, change τ by the automorphism 3. to get $P^\tau = \pm P$. If $P^\tau = -P$ change τ by the automorphism 4. to get $P^\tau = P$. Hence after changing τ by automorphisms on the list, we may assume $T^\tau = T$, $S^\tau = D_3SD_3$, $P^\tau = P$. But then τ is just an inner automorphism by a factor of D_3 , and therefore is on our list. This completes the proof for the present case.

CASE I(b). Suppose next that Y_3 is anti-diagonal, say

$$Y_3 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

After changing τ by an inner automorphism with factor $[u^{-1}, -v^{-1}, -v^{-1}]$, we may assume that $Y_1 = I$, $D_i^\tau = \pm D_i$ and $Y_3 = S_0$. Then $T^\tau = \pm S_0^{-1}T_0S_0 \dot{+} (1)$; the same type of argument as above shows that the minus sign is impossible. Hence $T^\tau = S_0^{-1}T_0S_0 \dot{+} (1) = T'^{-1}$, and we find again that either $U^\tau = U$ or $U^\tau = U_3$, whence $S^\tau = S$ or $S^\tau = D_3SD_3^{-1}$. Furthermore, we obtain $P^\tau = \pm P$ or

$\pm \bar{P}$ as before, and changing τ by 3. and 4. as needed, we get $P^r = P$. Now change τ by 2. Since $S = S'^{-1}$, $\bar{P} = P'^{-1}$ we find that $T^r = T$, $S^r = S$ or $D_3 S D_3$, $P^r = \bar{P}$ which is clearly a product of automorphisms on the list. We have completed the proof for this case.

CASE II. Suppose that in (5.4) each Y_r is anti-diagonal. After an inner automorphism by a suitably chosen diagonal matrix, we may assume that $Y_1 = Y_3 = S_0$. We then find that $T^r = (\pm T_0'^{-1}) \dot{+} (1)$, and the same reasoning as before shows that the minus sign cannot occur. Thence we obtain $(S_0 \dot{+} (1))^r = S_0 \dot{+} (1)$, and $S^r = S$ or $D_3 S D_3$. In the latter case, an inner automorphism by a factor of D_3 gives a new τ with $T^r = T'^{-1}$, $S^r = S$. Changing this τ by $X \rightarrow X'^{-1}$ we arrive at an automorphism τ which leaves U , S and T invariant. The same reasoning as in Case I(a) shows that $P^r = \pm P$ or $\pm \bar{P}$, and the remainder of the proof is as before.

6. **Generators of $\mathfrak{A}_n$.** We are now ready to prove the main theorem by induction on n .

We suppose $n \geq 4$, and that the result holds for $n-1$. Let D_j be the diagonal matrix $[1, \dots, 1, -1, 1, \dots, 1]$ with -1 occurring in the j th position. By Theorem 2.3, given any $\tau \in \mathfrak{A}_n$, we may change τ by an inner automorphism so as to achieve $D_1^r = \pm D_1$. Therefore τ maps $\mathfrak{C}(D_1) \cap G_n^+$ onto itself. Hence if $A \in G_{n-1}$ and $a \cdot \det A = 1$, we have

$$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}^\tau = \begin{pmatrix} \lambda_1(A) & 0 \\ 0 & A^{\sigma_1} \end{pmatrix}$$

where $\lambda_1: G_{n-1} \rightarrow G_1$ is a homomorphism and σ_1 is an automorphism of G_{n-1} . As before, $\lambda_1(a) = a$ always or $\bar{a}$ always. Using the induction hypothesis, we may write

$$A^{\sigma_1} = (\det A)^{m_1} Y_1 A^{\omega_1} Y_1^{-1}, \quad Y_1 \in G_{n-1},$$

where ω_1 is a product of automorphisms chosen from 2. or 3. After an inner automorphism with factor $(1) \dot{+} Y_1^{-1}$, we may take $Y_1 = I$. Now $D_1 D_2 = [-1, -1, 1, \dots, 1]$; by computing $(D_1 D_2)^r$ we find that $D_2^r = \pm D_2$. Likewise $D_r^r = \pm D_r$, $1 \leq r \leq n$. We may therefore write

$$(6.1) \quad ((a) + {}^r A)^\tau = \lambda_r(A) + {}^r (\det A)^{m_r} Y_r A^{\omega_r} Y_r^{-1},$$

where $A \in G_{n-1}$ is arbitrary, $a \cdot \det A = 1$, $\lambda_r: G_{n-1} \rightarrow G_1$ is a homomorphism such that either $\lambda_r(A) = a$ always or $\bar{a}$ always, where $Y_r \in G_{n-1}$, and ω_r is a product of some (of none) of the automorphisms 2., 3. (Further, we have already seen that we may choose $Y_1 = I$.)

Now let $Z \in G_{n-2}$; since $(I^{(2)} \dot{+} Z) \in \mathfrak{C}(D_1) \cap \mathfrak{C}(D_2)$ we can compute $(I^{(2)} \dot{+} Z)^r$ in two ways. This gives

$$Y_1 \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} Y_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & Z_1 \end{pmatrix}$$

for some $Z_1 \in G_{n-2}$. Since such a relation holds for all $Z \in G_{n-2}$, it follows that $Y_1 = (y_1) \dot{+} \tilde{Y}$. By a similar argument we see that Y_1 , and indeed each Y_r , must be diagonal, and that all Y_r are sections of a single diagonal matrix D . Following τ with an inner automorphism with factor D^{-1} , we see that we may assume each $Y_r = I$. The same type of argument shows that the various m_r are all the same, and that all the ω_r coincide. Hence we have

$$X^\tau = X^\omega$$

for all decomposable $X \in G_n^+$, where ω is the common automorphism $\omega_1 = \omega_2 = \cdots$, i.e., ω is a product of automorphisms chosen from 2. or 3. Changing τ by the automorphisms 2., 3. as needed we may thus assume that $X^\tau = X$ for all decomposable $X \in G_n^+$. For $n \geq 4$, these decomposable matrices generate G_n^+ , and so $X^\tau = X$ for all $X \in G_n^+$.

We now determine the effect of τ on G_n^- and $G_n^{\pm i}$. Let $Y, Z \in G_n^-$ where Z is fixed. Then

$$Y^\tau Z^\tau = (YZ)^\tau$$

implies

$$Y^\tau = YB$$

for all $Y \in G_n^-$, where B is independent of Y . Using $(Y^2)^\tau = (Y^\tau)^2$, we obtain

$$BYB = Y$$

for all $Y \in G_n^-$. This implies that $B = \pm I$ or $\pm iI$. However, $B = \pm iI$ is impossible, and therefore $B = \pm I$, whence

$$Y^\tau = \pm Y$$

for all $Y \in G_n^-$. If n is odd, $\tau: G_n^- \rightarrow G_n^-$ shows that only the plus sign can hold. If n is even, then changing τ by the automorphism $X \rightarrow (\det X)X$, if necessary, we may assume that $X^\tau = X$ for all $X \in G_n^+ \cup G_n^-$.

The same argument as above shows that $Y^\tau = \pm Y$ for all $Y \in G_n^{\pm i}$. If the plus sign occurs, τ is the identity; if the minus sign occurs, then τ is simply the automorphism $X^\tau = (\det X)^2 X$. This concludes the proof of the Main Theorem.

Another approach to the proof of the Main Theorem, which is less computational than that given here, is contained in references [3] and [4].

REFERENCES

1. L. K. Hua and I. Reiner, *On the generators of the symplectic modular group*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 415-426.
2. ———, *Automorphisms of the unimodular group*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 331-348.
3. J. Landin and I. Reiner, *Automorphisms of the general linear group over a principal ideal domain*, Ann. of Math. vol. 65 (1957) pp. 519-526.

4. ———, *Automorphisms of the two-dimensional general linear group over a euclidean ring*, to appear in Proc. Amer. Math. Soc.
5. G. W. Mackey, *Isomorphisms of normed linear spaces*, Ann. of Math. vol. 43 (1942) pp. 244–260.
6. I. Reiner, *Maximal sets of involutions*, Trans. Amer. Math. Soc. vol. 79 (1955) pp. 459–476.
7. C. E. Rickart, *Isomorphic groups of linear transformations*, Amer. J. Math. vol. 72 (1950) pp. 451–464.

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